Arab J Math (2013) 2:209–219
DOI 10.1007/s40065-013-0066-1

Yekini Shehu

Modified Krasnoselskii–Mann iterative algorithm for nonexpansive mappings in Banach spaces

Received: 27 June 2012 / Accepted: 5 January 2013 / Published online: 26 January 2013

Abstract In this paper, we prove that the sequence $\{x_n\}$ generated by modified Krasnoselskii–Mann iterative algorithm introduced by Yao et al. [J Appl Math Comput 29:383–389, 2009] converges strongly to a fixed point of a nonexpansive mapping T in a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. Furthermore, we present an example that illustrates our result in the setting of a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. The results of this paper extend and improve several results presented in the literature in the recent past.

Mathematics Subject Classification 47H09 · 47J25

الملخص

في هذه الورقة، نبرهن أن المتتالية $\{x_n\}$ المولدة بواسطة خوارزمية كراسنوسلكيي – مان التكرارية المعدلة والتي عرضها ياو ورفقاؤه سنة 2009، تقترب بقوة إلى نقطة ثابتة لراسم غير توسعي T، معرف على فضاء بناخ حقيقي ومحدب بانتظام وبمعيار قابل لتفاضل جاطو (Gâteaux) المنتظم. بالإضافة إلى ذلك، نعرض مثالاً يوضح نتيجتنا في حالة فضاء بناخ حقيقي ومحدب بانتظام، وبمعيار قابل لتفاضل جاطو (Gâteaux) المنتظم. إن نتائج هذا العمل توسع وتطور عدة نتائج أخرى عرضت مؤخراً في أعمال سابقة.

1 Introduction

Let E be a real Banach space and C a nonempty, closed and convex subset of E. We denote by J the normalized duality map from E to 2^{E^*} (E^* is the dual space of E) defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$. We denote by $F(T) = \{x \in C : Tx = x\}$ the set of fixed points of T. It is assumed throughout that $F(T) \neq \emptyset$.

One of the most important fixed point theorems and application is the classical contraction mapping principle, or, in other words, the Banach–Caccciopli fixed point theorem which is the following

Theorem 1.1 (Banach [2], Cacciopoli [14]) *Let (X, ρ) be a complete metric space and let $T : (X, \rho) \to (X, \rho)$ satisfy*

$$\rho(T(x), T(y)) \leq \gamma\rho(x, y) \tag{1.1}$$

Y. Shehu (✉)
Department of Mathematics, University of Nigeria, Nsukka, Nigeria
E-mail: deltanougt2006@yahoo.com

for some non-negative constant $\gamma < 1$ and for each $x, y \in X$. Then T has a unique fixed point in X. Moreover, starting with arbitrary $x_0 \in X$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \quad n \geq 0, \tag{1.2}$$

converges strongly to the unique fixed point.

The iterative technique (1.2) is due to Picard [30]. A mapping T satisfying (1.1) is called a *strict contraction*. If $\gamma = 1$ in the relation (1.1), then T is called nonexpansive. For the iterative formula, it was observed that if the condition $\gamma < 1$ on the operator T is weakened to $\gamma = 1$, the sequence $\{x_n\}$ defined by (1.2) may fail to converge to a fixed point of T. This can be seen by considering an anticlockwise rotation of the unite disc of $\mathbb{R}^2$ about the origin through an angle of say, $\frac{\pi}{4}$. This map is nonexpansive, but the Picard sequence fails to converge. Krasnoselskii [23], however, showed that in this example, if the Picard iteration formula is replaced by the following formula

$$x_0 \in X, \qquad x_{n+1} = \frac{1}{2}(x_n + Tx_n), \quad n \geq 0,$$

then the iterative sequence converges to the fixed point.

However, in 1953, the most general iterative formula for approximation of fixed points of nonexpansive mapping which is called Krasnoselskii–Mann iterative algorithm was introduced (in the light of [26]) as follows:

$$x_0 \in K, \qquad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}$ is a real sequence in the interval (0,1). Though simple in form, the Krasnoselskii–Mann iteration is remarkably useful for finding fixed points of a nonexpansive mapping and provides a unified framework for many algorithms from various different fields. In this respect, the following result is basic and important.

Theorem 1.2 (Genel and Lindenstrass [19]) *Let T be a nonexpansive mapping on a real Hilbert space H. Then the sequence $\{x_n\}$ defined by (1.3) converges weakly to a fixed point of T, provided $\alpha_n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$, whenever each fixed point exists.*

However, we note that all results in the literature on the Krasnoselskii–Mann iterative algorithm for nonexpansive mappings have only weak convergence even in a real Hilbert space. For more details, please see [19]. Yang and Zhao [36,42] further generalized the Krasnoselskii–Mann iteration and proposed the generalized KM theorems. For the details, please see [36,42].

Apart from being an obvious generalization of the contraction mappings, nonexpansive mappings are important, as has been observed by Bruck [11], mainly for the following two reasons:

1. Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960s and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the fine geometric properties of the underlying Banach spaces instead of compactness properties.
2. Nonexpansive mappings appear in applications as the transition operators for initial valued problems of differential inclusions of the form $0 \in \frac{du}{dt} + T(t)u$, where the operators $\{T(t)\}$ are, in general, set-valued and are *accretive* or *dissipative* and *minimally continuous*.

Construction of fixed points of nonexpansive mappings is an important subject in nonlinear mapping theory and its applications; in particular, in image recovery and signal processing (see, e.g., [13,31,38]). For the past 30 years or so, the study of Krasnoselskii–Mann iterative procedures for the approximation of fixed points of nonexpansive mappings and fixed points of some of their generalizations and approximation of zeros of accretive-type operators have been a flourishing area of research for many mathematicians. For example, the reader can consult the recent monographs of Berinde [3] and Chidume [16].

Another iterative scheme which is more general than the Krasnoselskii–Mann iterative procedures and which converges to a fixed point of Lipschitz pseudocontractive self mapping T of K is the Ishikawa iterative scheme (introduced by Ishikawa [21]). Ishikawa [21] proved the following theorem.

Theorem 1.3 *Let K be a compact convex subset of a Hilbert space H, $T : K \to K$ a Lipschitz pseudocontractive mapping and x_0 any point of K. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each integer $n \geq 0$ by*

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T x_n \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T yn, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences of positive numbers satisfying the conditions*
(i) $0 \leq \alpha_n \leq \beta_n < 1$; (ii) $\lim_{n\to\infty} \beta_n = 0$,
(iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n = +\infty$.

The Ishikawa iteration scheme has been extensively studied by many authors (see, e.g., [33,41,43] and the references therein).

Recently, Yao et al. [37] introduced a modified Krasnoselskii–Mann iterative algorithm for nonexpansive mappings in the framework of a real Hilbert space and proved the following theorem.

Theorem 1.4 (Yao et al. [37]) *Let H be a real Hilbert space. Let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For given $x_0 \in H$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by*

$$\begin{cases} y_n = (1-\alpha_n)x_n; \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \quad n \geq 0, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in [0,1] which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C3) $\beta_n \in [a, b] \subset (0, 1)$.

Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to a point in T.

The purpose of this paper is to prove a strong convergence theorem for approximation of fixed point of a nonexpansive mapping using a modified Krasnoselskii–Mann iterative algorithm introduced by Yao et al. [37] in a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. Our theorem extends Theorem 1.4 from Hilbert spaces to a more general uniformly convex Banach space with uniformly Gâteaux differentiable norm.

2 Preliminaries

Let E be a real normed linear space and let $S := \{x \in E : \|x\| = 1\}$. E is said to have a *Gâteaux differentiable* norm if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. When this limit exists, we say that E is smooth. E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Furthermore, E is said to be uniformly smooth if the limit exists uniformly for $(x, y) \in S \times S$. It is known that if E is smooth then any duality mapping on E is single-valued; and if E has a uniformly Gâteaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of E.

Let C be a nonempty, closed, convex and bounded subset of a Banach space E and let the diameter of C be defined by $d(C) := \sup\{\|x - y\| : x, y \in C\}$. For each $x \in C$, let $r(x, C) := \sup\{\|x - y\| : y \in C\}$ and let $r(C) := \inf\{r(x, C) : x \in C\}$ denote the Chebyshev radius of C relative to itself. The *normal structure coefficient* $N(E)$ of E (introduced by Bynum [12], see also Lim [24] and the references contained therein) is defined by $N(E) := \inf\{\frac{\mathrm{d}(C)}{r(C)} : C$ is a closed, convex and bounded subset of E with $\mathrm{d}(C) > 0\}$. A space E such that $N(E) > 1$ is said to have *uniform normal structure*. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [16,25]).

In the sequel, we shall also make use of the following lemmas.

Lemma 2.1 *Let E be a real normed linear space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle \;\forall\, x, y \in E, \;\forall\, j(x + y) \in J(x + y).$$

Lemma 2.2 (Xu [35], Zalinescu [39,40]) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \le r\}$ and $\lambda \in [0, 1]$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R}, \quad g(0) = 0$$

such that for every $x, y \in B_r(0)$, the following inequality holds:

$$||\lambda x + (1-\lambda)y||^2 \le \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||).$$

Lemma 2.3 (Browder [10], Goebel and Kirk [20]) *Let E be a uniformly convex Banach space, C a closed convex subset of E, and $T : C \to C$ a nonexpansive mapping with a fixed point. Assume that a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to y$. Then $x - Tx = y$.*

Lemma 2.4 (Lim and Xu [25]) *Suppose E is a Banach space with uniform normal structure, C a nonempty bounded subset of E and $T : C \to C$ is a uniformly L-Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$. Suppose also there exists a nonempty bounded closed convex subset K of C with the following property (P):*

$$x \in K \textit{ implies } \omega_w(x) \in K,$$

(where $\omega_w(x)$ is the ω-limit set of T at x, that is, the set $\{y \in E : y = \text{ weak } \omega - \lim T^{n_j}x \text{ for some } n_j \to \infty\}$). Then T has a fixed point in K.

Lemma 2.5 (Shioji and Takahashi [32]) *Let $(x_0, x_1, x_2, ...) \in l_\infty$ be such that $\mu_n x_n \le 0$ for all Banach limits μ. If $\limsup_{n\to\infty}(x_{n+1} - x_n) \le 0$, then $\limsup_{n\to\infty} x_n \le 0$.*

Lemma 2.6 (Xu [34]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \ge 0,$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\lim sup\ \sigma_n \le 0$; (iii) $\gamma_n \ge 0$; $(n \ge 0)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

3 Main results

We next prove the following strong convergence theorem using modified Krasnoselskii–Mann iterative algorithm for nonexpansive mappings.

Theorem 3.1 *Let E be a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. Let $T : E \to E$ be a nonexpansive mapping such that $F(T) \ne \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by $x_1 \in E$,*

$$\begin{cases} y_n = (1-\alpha_n)x_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n Ty_n, \quad n \ge 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in [0,1] which satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\beta_n \in [a, b] \subset (0, 1)$.

Then the sequence $\{x_n\}$ converges strongly to a point in $F(T)$.

Proof First we show that $\{x_n\}$ is bounded. For any $p \in F(T)$, from (3.1) we have

$$\begin{aligned} \|x_{n+1} - p\| &\le (1-\beta_n)\|y_n - p\| + \beta_n\|Ty_n - p\| \\ &\le \|y_n - p\| \\ &= \|(1-\alpha_n)x_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|p\| \\ &\le \max\{\|x_n - p\|, \|p\|\}. \end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \le \max\{\|x_1 - p\|, \|p\|\}, \quad \forall n \ge 1.$$

Hence, $\{x_n\}$ is bounded and also are $\{y_n\}$ and $\{Ty_n\}$.
Using Lemma 2.2 and (3.1), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||(1-\beta_n)(y_n - p) + \beta_n(Ty_n - p)||^2 \\ &\le (1-\beta_n)||y_n - p||^2 + \beta_n||Ty_n - p||^2 - \beta_n(1-\beta_n)g(||Ty_n - y_n||) \\ &\le (1-\beta_n)||y_n - p||^2 + \beta_n||y_n - p||^2 - \beta_n(1-\beta_n)g(||Ty_n - y_n||) \\ &= ||y_n - p||^2 - \beta_n(1-\beta_n)g(||Ty_n - y_n||). \end{aligned}$$

Therefore, by Lemma 2.1, we have

$$\begin{aligned} \beta_n(1-\beta_n)g(||Ty_n - y_n||) &\le ||y_n - p||^2 - ||x_{n+1} - p||^2 \\ &\le ||x_n - p||^2 - ||x_{n+1} - p||^2 + 2\alpha_n\langle x_n - p, j(y_n - p)\rangle. \end{aligned} \tag{3.2}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then there exists a constant $M \ge 0$ such that

$$\langle x_n - p, j(y_n - p)\rangle \le M \quad \text{for all} \quad n \ge 1.$$

So, from (3.2) we have

$$\beta_n(1-\beta_n)g(||Ty_n - y_n||) \le ||x_n - p||^2 - ||x_{n+1} - p||^2 + 2\alpha_n M. \tag{3.3}$$

To prove that $\{x_n\}$ converges to p, we have two cases.

Case 1 Assume that the sequence $\{||x_n - p||\}$ is monotonically decreasing sequence. Then $\{||x_n - p||\}$ is convergent. Clearly, we have

$$||x_{n+1} - p||^2 - ||x_n - p||^2 \to 0.$$

It then implies from (3.3) that

$$\lim_{n\to\infty} \beta_n(1-\beta_n)g(||Ty_n - y_n||) = 0.$$

Using the condition $\beta_n \in [a, b] \subset (0, 1)$ and the property of g, we have

$$\lim_{n\to\infty} ||Ty_n - y_n|| = 0. \tag{3.4}$$

Now from (3.4), we obtain

$$||y_n - x_{n+1}|| = \beta_n||Ty_n - y_n|| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}$$

From (3.1), we know that

$$||y_n - x_n|| = \alpha_n||x_n|| \to 0. \tag{3.6}$$

Therefore, from (3.6) and (3.5), we have

$$||x_{n+1} - x_n|| \to 0 \quad \text{as} \quad n \to \infty.$$

Also, from (3.4) and (3.6), we have

$$\begin{aligned} ||Tx_n - x_n|| &\le ||Tx_n - Ty_n|| + ||Ty_n - y_n|| + ||y_n - x_n|| \\ &\le 2||x_n - y_n|| + ||Ty_n - y_n|| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.7}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $p \in E$. Furthermore, by Lemma 2.3, we have $p \in F(T)$.

We now prove that

$$\limsup_{n\to\infty}\langle -p, j(y_n-p)\rangle \le 0.$$

Define a map $\phi : E \to \mathbb{R}$ by

$$\phi(x) := \mu_n\|y_n - x\|^2, \quad \forall x \in E.$$

Then, $\phi(x) \to \infty$ as $\|x\| \to \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $y^* \in E$ such that $\phi(y^*) = \min_{u\in E}\phi(u)$. Hence, the set

$$K^* := \left\{x \in E : \phi(x) = \min_{u\in E}\phi(u)\right\} \neq \emptyset.$$

We shall make use of Lemma 2.4. If $x \in K^*$ and $y := \omega{-}\lim_{j\to\infty} T^{m_j}x$, then from weak lower semi-continuity of ϕ and $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$, we have (since $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$ implies $\lim_{n\to\infty}\|y_n - T^m y_n\| = 0, \quad m \ge 1$, this is easily proved by induction),

$$\begin{aligned}
\phi(y) &\le \liminf_{j\to\infty}\phi\left(T^{m_j}x\right) \le \limsup_{m\to\infty}\phi\left(T^m x\right)\\
&= \limsup_{m\to\infty}\left(\mu_n\|y_n - T^m x\|^2\right)\\
&= \limsup_{m\to\infty}\left(\mu_n\|y_n - T^m y_n + T^m y_n - T^m x\|^2\right)\\
&\le \limsup_{m\to\infty}\left(\mu_n\|T^m y_n - T^m x\|^2\right)\\
&\le \limsup_{m\to\infty}\left(\mu_n\|y_n - x\|^2\right)\\
&= \phi(x) = \inf_{u\in E}\phi(u).
\end{aligned}$$

So, $y^* \in K^*$. By Lemma 2.4, T has a fixed point in K^* and so $K^* \cap F(T) \neq \emptyset$. Without loss of generality, assume that $y^* = p \in K^* \cap F(T)$. Let $t \in (0, 1)$. Then, it follows that $\phi(p) \le \phi(p - tp)$ and using Lemma 2.1, we obtain that

$$\|y_n - p + tp\|^2 \le \|y_n - p\|^2 + 2t\langle p, j(y_n - p + tp)\rangle$$

which implies that

$$\mu_n\langle -p, j(y_n - p + tp)\rangle \le 0.$$

Moreover,

$$\begin{aligned}
\mu_n\langle -p, j(y_n-p)\rangle &= \mu_n\langle -p, j(y_n-p) - j(y_n-p+tp)\rangle\\
&\quad+\mu_n\langle -p, j(y_n-p+tp)\rangle\\
&\le \mu_n\langle -p, j(y_n-p) - j(y_n-p+tp)\rangle.
\end{aligned}$$

Furthermore, the fact that the normalized duality mapping is norm-to-$weak^*$ uniformly continuous on bounded subsets of E gives, as $t \to 0$ with n fixed, that

$$\langle -p, j(y_n-p)\rangle - \langle -p, j(y_n-p+tp)\rangle \to 0.$$

Thus, for all $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that for all $t \in (0, \delta_\epsilon)$ and for all $n \ge 1$,

$$\langle -p, j(y_n-p)\rangle - \langle -p, j(y_n-p+tp)\rangle < \epsilon.$$

Thus,

$$\mu_n\langle -p, j(y_n-p)\rangle - \mu_n\langle -p, j(y_n-p+tp)\rangle \le \epsilon,$$

which implies (since $\epsilon > 0$ is arbitrary) that

$$\mu_n\langle -p, j(y_n - p)\rangle \leq 0.$$

Observe that

$$||y_{n+1} - y_n|| \leq ||y_{n+1} - x_{n+1}|| + ||x_{n+1} - y_n||.$$

Then by (3.5) and (3.6), we have

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0.$$

Again, using the fact that the normalized duality mapping j is norm-to-weak* uniformly continuous on bounded subsets of E, we have that

$$\lim_{n\to\infty}\Big[\langle -p, j(y_n - p)\rangle - \langle -p, j(y_{n+1} - p)\rangle\Big] = 0$$

and so we obtain by Lemma 2.5 that

$$\limsup_{n\to\infty}\langle -p, j(y_n - p)\rangle \leq 0.$$

Finally, from the recursion formula (3.1) and Lemma 2.1, we have the following:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\beta_n)(y_n - p) + \beta_n(Ty_n - p)\|^2\\
&\leq (1-\beta_n)\|y_n - p\|^2 + \beta_n\|Ty_n - p\|^2\\
&\leq \|y_n - p\|^2\\
&= \|(1-\alpha_n)x_n - p\|^2\\
&= \|(1-\alpha_n)(x_n - p) - \alpha_n p\|^2\\
&\leq (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle -p, j(y_n - p)\rangle.
\end{aligned}$$

By using Lemma 2.6, we have that $\{x_n\}$ converges strongly to $p \in F(T)$.

Case 2 Assume that $\{||x_n - p||\}$ is not monotonically decreasing sequence. Set $\Gamma_n = ||x_n - p||^2$ and let $\tau : N \to N$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in N : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for $n \geq n_0$. From (3.3), we see that

$$\beta_{\tau(n)}(1-\beta_{\tau(n)})g(||Ty_{\tau(n)} - y_{\tau(n)}||) \leq 2\alpha_{\tau(n)}M \to 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, we have

$$||Ty_{\tau(n)} - y_{\tau(n)}|| \to 0 \quad \text{as} \quad n \to \infty.$$

By the same argument as in Case 1, we can show that $x_{\tau(n)}$ converges weakly to p as $\tau(n) \to \infty$ and $\limsup_{\tau(n)\to\infty}\langle -p, j(y_{\tau(n)} - p)\rangle \leq 0$. We know that for all $n \geq n_0$,

$$0 \leq ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2 \leq 2\alpha_n[\langle -p, j(y_{\tau(n)} - p)\rangle - ||x_{\tau(n)} - p||^2],$$

which implies that

$$||x_{\tau(n)} - p||^2 \leq \langle -p, j(y_{\tau(n)} - p)\rangle.$$

Then we conclude that

$$\lim_{n\to\infty} ||x_{\tau(n)} - p|| = 0.$$

Therefore

$$\lim_{n\to\infty}\Gamma_{\tau(n)}=\lim_{n\to\infty}\Gamma_{\tau(n)+1}=0.$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim_{n\to\infty}\Gamma_n = 0$, that is, $\{x_n\}$ converges strongly to p. Consequently, it is easy to prove that $\{y_n\}$ converges strongly to p. This complete the proof. □

Remark 3.2 Our Theorem 3.1 extended the results of Yao et al. [37] from *real Hilbert space* to *real uniformly convex Banach space with uniformly Gâteaux differentiable norm*.

We note that every uniformly smooth Banach space has uniformly Gâteaux differentiable norm. Hence, our theorem is applicable in a uniformly convex Banach space which is also uniformly smooth.

Corollary 3.3 *Let E be a real uniformly convex Banach space which is also uniformly smooth. Let $T : E \to E$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by* (3.1)*. Then the sequence $\{x_n\}$ converges strongly to a point in $F(T)$.*

Remark 3.4 Furthermore, our results are also applicable in $(L_p)\ell_p,\ \ 1 < p < \infty$ and $W^{m,p},\ \ 1 < p < \infty$.

Remark 3.5 The problem of finding fixed points of nonexpansive mappings has attracted much attention because of its extraordinary utility and broad applicability in many branches of mathematical science and engineering.

Krasnoselskii–Mann's algorithm is a widely used method for solving a fixed point equation of the form $Tx = x$, where C is a nonempty, closed and convex subset of a Banach space E, and $T : C \to C$ is a nonexpansive mapping.

Krasnoselskii–Mann's algorithm converges weakly to a fixed point of T provided the underlying space E is a Hilbert space or more general, a uniformly convex Banach space which has a Fréchet differentiable norm or satisfies Opial's property. It is a very interesting topic of constructing some algorithms such that the strong convergence of proposed algorithms are guaranteed. For this purpose, in this section we present a modified Krasnoselskii–Mann method (3.1) for nonexpansive mappings in real uniformly convex Banach space with uniformly Gâteaux differentiable norm and show that the proposed method (3.1) has strong convergence. Furthermore, it is worth mentioning that our proof is different from those in the literature.

4 Application

Let E be a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. A mapping $A : D(A) \subseteq E \to E$ is called *accretive* if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq 0 \quad \text{for} \quad x, y \in D(A). \tag{4.1}$$

As a result of Kato [22], it follows from Inequality (4.1) that A is *accretive* if the following inequality

$$||x-y\| \leq \|x-y+s(Ax-Ay)|| \tag{4.2}$$

holds for every $x, y \in D(A)$ and for all $s > 0$. If E is a Hilbert space, accretive operators are also called *monotone*. An operator A is called *m-accretive* if it is accretive and $R(I+rA)$, the range of $(I+rA)$, is E for all $r > 0$; and A is said to satisfy the *range condition* if $cl(D(A)) \subseteq R(I+rA)$, for all $r > 0$, where $cl(D(A))$ denotes the closure of the domain of A. If A is m-accretive (see, e.g., [17]) then $J_A := (I+A)^{-1}$ is a nonexpansive single-valued mapping from $R(I+A)$ to $D(A)$ and $F(J_A) = N(A)$, where $N(A) := \{x \in D(A) : Ax = 0\}$ and $F(J_A) := \{x \in E : J_A x = x\}$.

The accretive operators were introduced independently in 1967 by Browder [4] and Kato [22]. Interest in such mappings stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in Banach spaces. It is well known that many physically significant problems can be modeled in terms of an initial value problem of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0, u(0) = u_0, \tag{4.3}$$

where A is an accretive mapping on an appropriate Banach space. Typical examples of such evolution equations are found in models involving heat, wave or Schrödinger equation (see, e.g., [5]). An early fundamental result in the theory of accretive operators, due to Browder [4], states that the initial value problem (4.3) has a solution if A is locally Lipschitzian and accretive on E. Utilizing the existence result for (4.3), Browder [4] proved that if A is locally Lipschitzian and accretive on E, then A is m-accretive. A consequence of this is that the equation

$$Au = 0 \tag{4.4}$$

has a solution. Ray [29] gave an elementary and elegant proof of this result of Browder by using a fixed point theorem of Caristi [15]. Martin [27,28] proved that (4.3) is solvable if A is continuous and accretive on the space E; and utilizing this result, he further proved that if A is continuous and accretive, then A is m-accretive, thus generalizing the result of Browder [4]. Other existence theorems for zeros of accretive operators can be found in Browder [6–9]. See also Theorem 13.1 of Deimling [18].

We remark that in the evolution Eq. (4.3), if u is independent of t, then $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$ and the Eq. (4.3) reduces to (4.4) whose solution describes the equilibrium state or the stable state of the system described by (4.3). This is very important in many applications such as ecology, economics, physics, to name but a few. Consequently, considerable research efforts have been devoted to methods of solving Eq. (4.4) when A is accretive. Since generally A is nonlinear, there is no known method to obtain a closed form solution for this equation. As a result of this, the study of fixed point theory and iterative approximation of zeros of m-accretive mappings has attracted the interest of numerous scientists and has become a flourishing area of research for numerous mathematicians.

In this section, we describe applications of the previous results to finding zeros of accretive mappings. Precisely, let E be a real uniformly convex Banach space with uniformly Gâteaux differentiable norm and let $A : E \to E$ be a continuous and accretive mapping and consider the problem:

$$\text{find } x \in E \text{ such that } Ax = 0 \tag{4.5}$$

to which a solution is supposed to exist.

Theorem 4.1 *Let E be a real uniformly convex Banach space with uniformly Gâteaux differentiable norm. Let $A : E \to E$ be a continuous and accretive mapping such that $N(A) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by $x_1 \in E$,*

$$\begin{cases} y_n = (1 - \alpha_n)x_n, \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n J_A y_n, \quad n \geq 1, \end{cases} \tag{4.6}$$

where $J_A := (I + A)^{-1}$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] which satisfy the following conditions:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\beta_n \in [a, b] \subset (0, 1)$.

Then the sequence $\{x_n\}$ converges strongly to a point in $N(A)$.

Proof By the result of Cioranescu [17] and Martin [27,28], A is m-accretive and this implies that $J_A = (I + A)^{-1}$ nonexpansive and $F(J_A) = N(A)$. Taking $J_A = T$ in Theorem 3.1 and repeating the line of arguments of Theorem 3.1, we obtain the desired result. □

Remark 4.2 Our iterative scheme (3.1) is closely related to Ishikawa iterative scheme (1.4). Therefore, our results complement the results of Tan and Xu [33] and other results on approximation of fixed points of nonexpansive mappings using Ishikawa iterative scheme (1.4) in Banach spaces.

Remark 4.3 Our results in this paper complement the results of [1] and the results of Chapter 16 of Chidume [16]. Furthermore, our results extend the results of Theorem 5.4 of Berinde [3].

Remark 4.4 The problem of finding zeros of accretive mappings in a real uniformly convex Banach space with uniformly Gâteaux differentiable norm given in (4.5) above gave us the motivation to extend the result of Yao [37] from Hilbert spaces to real uniformly convex Banach spaces with uniformly Gâteaux differentiable norm.

Acknowledgments The author would like to express his sincere thanks to the editor and anonymous referees for their valuable suggestions and comments which improved the original version of the manuscript greatly.

References

1. Aoyama, K.; Kimura, Y.; Takahashi, W.; Toyoda, M.: Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 2350–2360 (2006)
2. Banach, S.: Sur les operations dans les ensemble abstraits et leur applications aux equations integrales. Fund. Math. **3**, 133–181 (1922)
3. Berinde, V.: Iterative approximation of fixed points. Lecture Notes in Mathematics 1912, Springer, Berlin 2007.
4. Browder, F.E.: Nonlinear mappings of nonexpansive and accretive type in Banach spaces. Bull. Am. Math. Soc. **73**, 875–882 (1967)
5. Browder. F.E.: Nonlinear operators and nonlinear equations of evolution in Banach spaces. In: Proceedings of Symposia in Pure Mathematics, vol. XVIII, No. Part 2, 1976
6. Browder, F.E.: The solvability of nonlinear functional equations. Duke Math. J. **30**, 557–566 (1963)
7. Browder, F.E.: Nonlinear elliptic boundary value problems. Bull. Am. Math. Soc. **69**, 862–874 (1963)
8. Browder, F.E.: Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces. Bull. Am. Math. Soc. **73**, 470–475 (1967)
9. Browder, F.E.: Nonlinear monotone and accretive operators in Banach spaces. Proc. Natl. Acad. Sci. USA **61**, 388–393 (1968)
10. Browder, F.E.: Nonexpansive nonlinear mappings in a Banach space. Proc. Nat. Acad. Sci. USA. **54**, 1041–1044 (1965)
11. Bruck, R.E.: Asymptotic behaviour of nonexpansive mappings. In: Sine, R.C. (ed.) Contemporary Mathematics, vol. 18. Fixed Points and Nonexpansive Mappings, AMS, Providence (1980)
12. Bynum, W.L.: Normal structure coefficients for Banach spaces. Pac. J. Math. **86**, 427–436 (1980)
13. Byrne, C.: Unified treatment of some algorithms in signal processing and image construction. Inverse Problems **20**, 103–120 (2004)
14. Caccioppoli, R.: Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale. Rend. Lincei **11**, 794–799 (1930)
15. Caristi, J.: The fixed point theory for mappings satisfying inwardness conditions. Ph.D. Thesis, The University of Iowa, Iowa City (1975)
16. Chidume, C.E.: Geometric properties of Banach spaces and nonlinear iterations. Springer Verlag Series: Lecture Notes in Mathematics, vol. 1965 (2009), XVII, 326p, ISBN 978-1-84882-189-7
17. Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems. Kluwer Academic, Dordrecht (1990)
18. Deimling, K.: Nonlinear Functional Analysis. Springer, Berlin (1985)
19. Genel, A.; Lindenstrass J.: An example concerning fixed points. Isr. J. Math. **22**, 81–86 (1975)
20. Goebel, K.; Kirk, W.A.: Topics in metric fixed point theory. Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990)
21. Ishikawa, S.: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44** (1), 147–150 (1974)
22. Kato, T.: Nonlinear semigroups and evolution equations. J. Math. Soc. Japan **19**, 508–520 (1967)
23. Krasnoselskii, M.A.: Two remarks on the method of successive approximations (Russian). UspehiMat. Nauk **10**, 123–127 (1955)
24. Lim, T.C.: Characterization of normal structure. Proc. Am. Math. Soc. **43**, 313–319 (1974)
25. Lim, T.C.; Xu, H.K.: Fixed point theorems for asymptotically nonexpansive mappings. Nonlinear Anal. TMA **22**, 1345–1355 (1994)
26. Mann, W.R.: Mean value method in iteration. Proc. Am. Math. Soc **4**, 506–510 (1953)
27. Martin, R.H.Jr.: A global existence theorem for authonomous differential equations in Banach spaces. Proc. Am. Math. Soc. **26**, 307–314 (1970)
28. Martin, Jr., R.H.: Nonlinear Operators and Differential Equations in Banach Spaces. Interscience, New York (1976)
29. Ray, W.O.: An elementary proof of surjectivity for a class of accretive operators. Proc. Am. Math. Soc. **75**, 255–258 (1979)
30. Picard, E.: Memoire sur la theorie des equations aux derives partielles et la methode des approximations successives. J. Math. Pures et Appl. **6**, 145–210 (1890)
31. Podilchuk, C.I.; Mammone, R.J.: Image recovery by convex projections using a least-squares constraint. J. Opt. Soc. Am. **A7**, 517–521 (1990)
32. Shioji, S.; Takahashim, W.: Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces. Proc. Am. Math. Soc. **125**, 3641–3645 (1997)
33. Tan, K.K.; Xu, H.K.: Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301–308 (1993)

34. Xu, H.K.: Iterative algorithms for nonlinear operators. J. London Math. Soc. **66**(2), 240–256 (2002)
35. Xu, H.K.: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127–1138 (1991)
36. Yang, Q.; Zhao, J.: Generalized KM theorems and their applications. Inverse Problem **21**, 1971–1979 (2006)
37. Yao, Y.; Zhou, H.; Liou, Y.C.: Strong convergence of a modified Krasnoselskii–Mann iterative algorithm for nonexpansive mappings. J. Appl. Math. Comput. **29**, 383–389 (2009)
38. Youla, D.: On deterministic convergence of iterations of related projection mappings. J. Vis. Commun. Image Represent **1**, 12–20 (1990)
39. Zalinescu, C.: On uniformly convex functions. J. Math. Anal. Appl. **95**, 344–374 (1983)
40. Zalinescu, C.: Analysis in General Vector Spaces. World Scientific, River Edge (2002)
41. Zhang, S.: On the convergence problems of Ishikawa and Mann iteration processes with error for ϕ-pseudo contractive type mappings. Appl. Math. Mech. **21**, 1–10 (2000)
42. Zhao, J.; Yang, Q.: A note on the Krasnoselskii–Mann theorem and its generalizations. Inverse Problem **23**, 1011–1016 (2007)
43. Zegeye, H.; Shahzad, N.; Alghamdi, M.A.: Convergence of Ishikawa's iteration method for pseudocontractive mappings. Nonlinear Anal. **74**, 7304–7311 (2011)